Section 2.7

Limits at Infinity

(1) Graphs at Infinity

(2) Calculating Limits at Infinity

  (A) Vertical Asymptotes

  (B) Horizontal Asymptotes

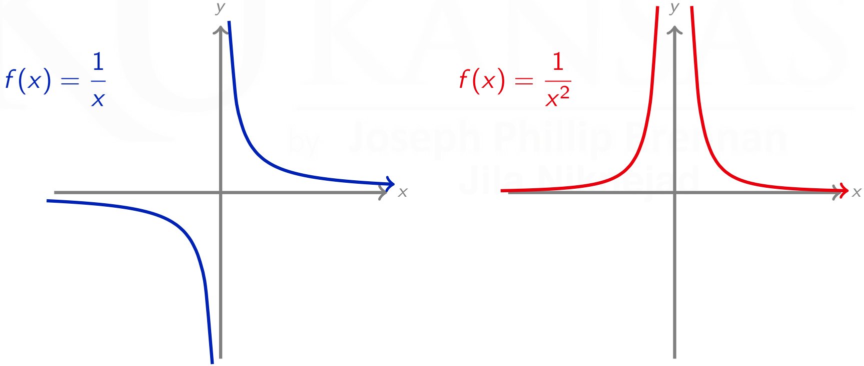
  (C) Roots and Infinite Limits

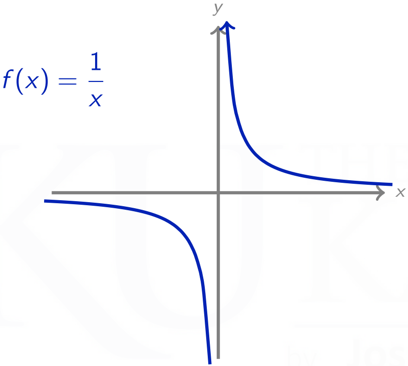
(3) Asymptotes of Common Functions

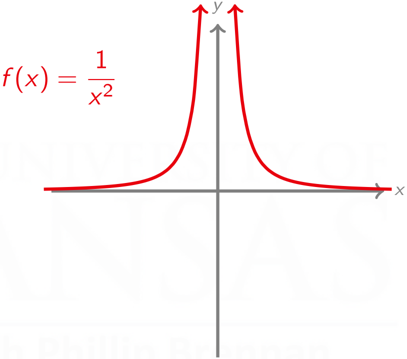
Infinity in the Input

$$\lim_{x\to\infty} f(x) = L$$

The values of $f(x)$ can be made as close to L as we would like by taking x sufficiently large.







Infinity in the Output

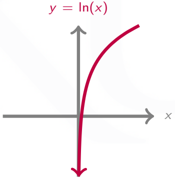
$$\lim_{x \to a} f(x) = \infty$$

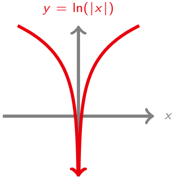
The values of $f(x)$ can be made as large as we want by taking x sufficiently close to a but not equal to a.

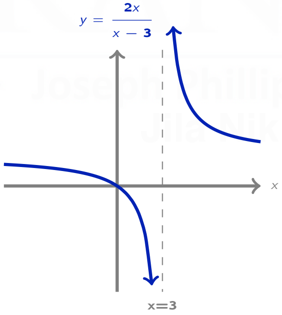
Vertical Asymptotes

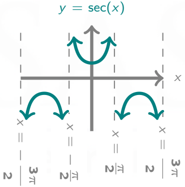
The line $x = a$ is a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

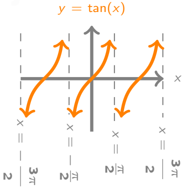
$$\lim_{x\to a} f(x) = \pm\infty \qquad \lim_{x\to a^+} f(x) = \pm\infty \qquad \lim_{x\to a^-} f(x) = \pm\infty$$











Horizontal Asymptotes

The line $y = L$ is a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

If n is a positive rational number, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad\qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

(if n has odd denominator, otherwise DNE)

Rational Functions:

$$\lim_{x\to\pm\infty} \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} = \begin{cases} 0 & \text{if } n < m \\ \dfrac{a_n}{b_n} & \text{if } n = m \\ \pm\infty & \text{if } n > m \end{cases}$$

Example I: Calculating Horizontal Asymptotes

(i) $$\lim_{x\to\infty} \frac{\sqrt{3+9x^6}}{1-2x^3} = \lim_{x\to\infty} \frac{|x^3|\sqrt{\frac{3}{x^6}+9}}{x^3(\frac{1}{x^3}-2)} = \frac{-3}{2}$$

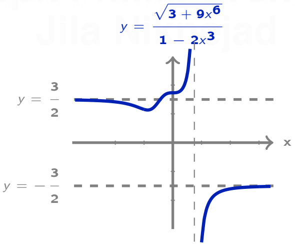
Reminder: $$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

(ii) $$\lim_{x\to-\infty} \frac{\sqrt{3+9x^6}}{1-2x^3} = \lim_{x\to-\infty} \frac{|x^3|\sqrt{\frac{3}{x^6}+9}}{x^3(\frac{1}{x^3}-2)} = \frac{3}{2}$$

Horizontal Asymptotes

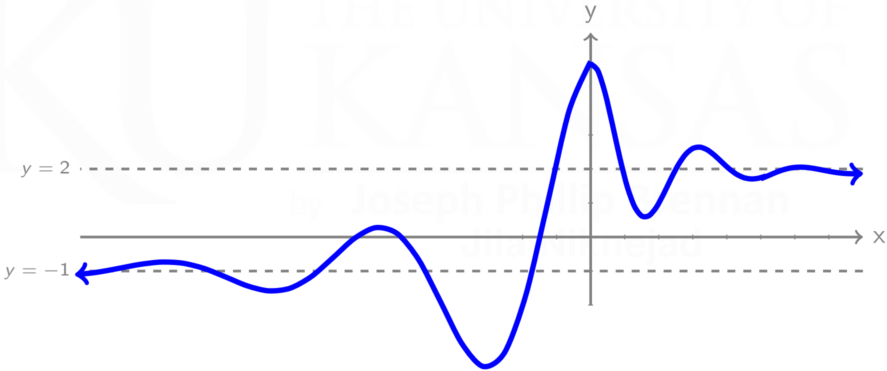
$$y = \frac{3}{2}$$

$$y = \frac{-3}{2}$$



The word **asymptote** always refers to behavior involving infinity.

Horizontal asymptotes refer to the **end behavior** of the graph. Functions can cross the lines of their horizontal asymptotes!



Example II: Calculating Horizontal Asymptotes

Find the Horizontal Asymptotes of the following functions:

(i) $f(x) = \sqrt{4x^2 + x} - 2x$

$\lim_{x \to -\infty} \sqrt{4x^2 + x} - 2x = \infty$ and $\lim_{x \to \infty} \sqrt{4x^2 + x} - 2x$ is of the form $\infty - \infty$.

$\lim_{x \to \infty} \sqrt{4x^2 + x} - 2x = \lim_{x \to \infty} \frac{(\sqrt{4x^2 + x} - 2x)(\sqrt{4x^2 + x} + 2x)}{\sqrt{4x^2 + x} + 2x} =$

$\lim_{x \to \infty} \frac{x}{\sqrt{4x^2 + x} + 2x} = \lim_{x \to \infty} \frac{x}{|x|(\sqrt{4 + \frac{1}{x}}) + 2x} = \lim_{x \to \infty} \frac{1}{(\sqrt{4 + \frac{1}{x}}) + 2} = \frac{1}{4}$. HA: $y = \frac{1}{4}$

(ii) $g(x) = x^3 - 2x^8$

$\lim_{x \to \pm\infty} (x^3 - 2x^8) = \lim_{x \to \pm\infty} x^8(\frac{1}{x^5} - 2) = \lim_{x \to \pm\infty} -2x^8 = -\infty$. HA: None

Video

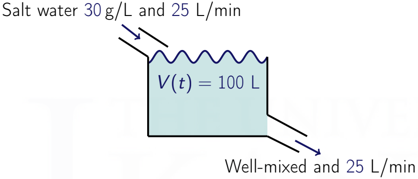
(iii) $h(x) = \arctan(e^x)$ Using the HA of $y = e^x$, $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to -\infty} e^x = 0$.

$\lim_{x \to \infty} \arctan(e^x) = \lim_{u \to \infty} \arctan(u) = \frac{\pi}{2}$ and $\lim_{x \to -\infty} \arctan(e^x) = \lim_{u \to 0} \arctan(u) = 0$.

HA: $y = 0$ and $y = \frac{\pi}{2}$

Video

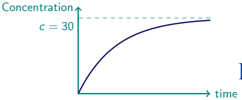
Example III: Applications of Horizontal Asymptotes



A tank contains 100 L of pure water. Brine that contains 30g of salt per liter of water is pumped into the tank at a rate of 25 $\frac{L}{min}$ and well-mixed solution exits at the same rate. What happens to the concentration of salt as time t approaches ∞?

As time goes to infinity, more and more of content of tank is being replaced by the water flowing in the tank. Therefore, the limit of concentration as time goes to infinity is the concentration of water flowing in: 30 $\frac{gr}{L}$.

Video



Summary of Limit Techniques

For finite limiting values we have four algebraic techniques:

(I) Direct Substitution *(Continuity and Determinate Forms)*

(II) Simplification

(III) Conjugation *(Roots)*

(IV) The Squeeze Theorem

For infinite limiting values we have two techniques:

(i) Manipulations involving $\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$

(ii) Horizontal asymptotes of common functions *($e^x, \arctan(x), \ldots$)*

Limits are at the heart of the Calculus sequence and will be used in defining every major object of study. Despite their role in defining objects, calculating limits becomes less common as we develop more techniques.

MATH125 has two remaining sections focused heavily on limits:

- Section 2.6/3.6 on trigonometric limits
- Section 4.5 on evaluating limits with indeterminate forms